Limitations of random multipliers in describing turbulent energy dissipation

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The intermittency of turbulent energy dissipation is often analyzed in terms of a product of random multipliers, each multiplier being the total dissipation in a subinterval r divided by the total dissipation in an interval λr. Recent experimental data of Pedrizzetti, Novikov, and Praskovsky [Phys. Rev. E **53**, 475 (1996)] give extensive information on the statistics of these multipliers. In this paper we further analyze these statistics, with emphasis on the universal scaling exponents. We emphasize that the scaling exponents are sensitive to the location of the subinterval within the interval and that they reflect dependence of successive multipliers. We show that these two sensitivities are related and strongly limit the direct applicability of multiplier statistics to the statistics of the turbulent energy dissipation. We extend Novikov's [Phys. Rev. E **51**, R3303 (1994)] ''gap theorem'' on the high moments of the multiplier to the case when the multipliers are statistically dependent and discuss its relevance to the high moments of the turbulent energy dissipation. [S1063-651X(96)09111-8]

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I. INTRODUCTION

The intermittency of the turbulent energy dissipation has been the object of intense study since Kolmogorov [1] first suggested that the relevant scaling variable is the local energy dissipation averaged over an inertial range interval of length r. He further proposed that this variable should have a log-normal distribution. Developments in this area until 1994 have been reviewed by Nelkin [2]. A log-normal distribution is a natural approximate consequence of a random multiplicative process. This connection was made explicit by Yaglom in 1966 [3], who introduced the multipliers as the ratio of the averaged dissipation in an interval r to the average dissipation in an interval λr, where $\lambda > 1$. In an important paper, Novikov [4] (see also [5,6]) developed a general framework for scale similarity of the random multipliers and pointed out that there are serious mathematical inconsistencies in assuming a limit log-normal distribution. Van Atta and Yeh [7], and later Chhabra and Sreenivasan [8], showed that scale similarity was approximately valid experimentally.

In all of the above works, it is either explicitly or implicitly assumed that the scaling properties of the random multipliers are essentially the same as the scaling properties of the averaged dissipation. It is frequently also assumed that the scaling properties of velocity differences in the inertial range can be simply related to the scaling properties of the averaged dissipation through Kolmogorov's refined similarity hypotheses [1]. In the present paper, we critically examine the connection between the multipliers and the dissipation. We have nothing new to say about the refined similarity hypotheses and thus draw no new conclusions about the scaling properties of velocity differences.

Recently there has been renewed interest in this subject. This was stimulated by Novikov's [9] proof that the asymptotic behavior of scaling exponents for high-order moments of the locally averaged energy dissipation rate is restricted if the probability density function (PDF) of the multiplier has no gap for large values corresponding to the most intense events. He suggested that the behavior of high moments in the popular She-Leveque model [10] is inconsistent with the absence of such a gap. Nelkin [11] pointed out that different models with totally different asymptotic behavior for scaling exponents cannot be distinguished experimentally.

Recently, Pedrizzetti, Novikov, and Praskovsky [12] (PNP) have investigated the statistical properties of the multipliers in much more detail than previously. The present paper is, in large part, a further analysis of the Pedrizzetti-Novikov-Praskovsky results. Our main conclusion is that the connection between the multipliers and the dissipation is complex, and that care has to be exercised when the scaling exponents derived from one are applied to the other. Thus, although Novikov's gap theorem is correct, it cannot be applied directly to the scaling exponents for the dissipation.

In Sec. II, we define the multipliers. In Sec. III we discuss the fact that the observed scaling exponents depend strongly on where the subinterval of size r is located within the interval of size λr. In Sec. IV, we examine deviations from scale similarity due to statistical dependence of the multipliers. We find that the Pedrizzetti-Novikov-Praskovsky data also gives large effects here. All of these considerations suggest that there is no simple connection between the statistics of the multipliers and the statistics of the dissipation. In Sec. V, we argue, however, that it is no accident that the case when $\lambda = 2$ and $\Delta = 1/2$ gives exponents closest to those observed for the dissipation. In Sec. VI, we look at higher moments, and the relevance of Novikov's gap theorem to high moments of the dissipation. Finally, in Sec. VII, we summarize our conclusions.

II. MULTIPLIERS AND ENERGY DISSIPATION

Consider a non-negative random field $\epsilon(x)$, which we call the rate of energy dissipation, but which is, in practice, the

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one-dimensional surrogate $\nu(\partial u/\partial x)^2$, where $u(x,t)$ is the instantaneous x component of the velocity. Introduce the total dissipation $E(r,x)$ in the interval of length r centered on x,

$$E(r,x)=r\epsilon_r(x)=\int_{x-r/2}^{x+r/2}\epsilon(s)ds. \quad (1)$$

Consider two segments of length r and λr, $\lambda>1$, embedded one inside the other, and define the multiplier as

$$M(r,\lambda,\Delta)=\frac{E(r,x)}{E(\lambda r,x')}\leqslant 1, \quad (2)$$

where the inequality in (2) derives from the non-negativity of the underlying field $\epsilon(x)$ and the parameter Δ, defined by

$$\Delta=\frac{x-x'}{\lambda r-r} \quad (3)$$

represents the relative displacement of the two segments. $\Delta=+1/2$ corresponds to the rightmost inner segment, while $\Delta=0$ means that the segments are centered at the same point. In the above we have followed the notation of PNP with the exception that their breakdown coefficient q is just λ times our multiplier M. If the statistics of the multiplier are universal, then they should be independent of r when r is an inertial range scale. The dependence of these statistics on the scale ratio λ and the inhomogeneity parameter Δ is the principal focus of this paper.

Perhaps the most important result in the theory of multipliers establishes [12] that a necessary and sufficient condition for the moments of M to scale as powers of λ (i.e., for scale similarity),

$$\langle[M(r,\lambda,\Delta)]^p\rangle=\lambda^{\gamma(p,\Delta)}, \quad (4)$$

is that (i) the PDF of $M(r,\lambda,\Delta)$, $P(M;r,\lambda,\Delta)$, is independent of r and (ii)

$$\langle M(r,\lambda_1\lambda_2,\Delta)^p\rangle=\langle M(r,\lambda_1,\Delta)^p\rangle\langle M(r\lambda_1,\lambda_2,\Delta)^p\rangle. \quad (5)$$

In Eq. (4),

$$\gamma(p,\Delta)=\mu(p,\Delta)-p \quad (6)$$

and the exponents $\mu(p,\Delta)$ are those defined by PNP. If strict scale similarity applies, then the exponents $\gamma(p,\Delta)$ and $\mu(p,\Delta)$ defined by (4) and (6) should be independent of the scale ratio. We consider this question further in Sec. IV.

If conditions (i) and (ii) above hold and the fluctuations in the dissipation averaged over a large scale L are relatively small (a reasonable and customary assumption), then one expects that the scaling exponents defined by the multipliers and those defined by the dissipation should be the same. As the scaling exponents for the dissipation must be independent of Δ, we conclude that under these assumptions $\mu(p,\Delta)$ should be independent of Δ. Let us justify this last statement. In each realization of an ensemble of turbulent flows, we measure $E(x,r)$ at a fixed x and all the multipliers having a fixed Δ and λ that connect a mother interval (fixed in space) at scale L with the daughter interval on which $E(x,r)$ is defined. Each of the $E(x,r)$ participating in the ensemble will have been produced by an array of multipliers

$$\frac{E(r)}{E(L)}=M(L/\lambda,\Delta)M(L/\lambda^2,\Delta)\cdots M(L/\lambda^n,\Delta). \quad (7)$$

Raising Eq. (7) to the power p and taking averages over the ensemble, one finds

$$\frac{\langle E(r)^p\rangle}{E(L)^p}=\langle M(L/\lambda,\Delta)^p M(L/\lambda^2,\Delta)^p\cdots M(L/\lambda^n,\Delta)^p\rangle. \quad (8)$$

Equation (8) is correct by construction and entails no contradiction. If now we *assume* the validity of conditions (i) and (ii), then, as Novikov has shown, the scaling of the moments of M is ensured, that is, $\langle M(\lambda,\Delta)^p\rangle=\lambda^{\gamma(p,\Delta)}$, and we are led to the conclusion that

$$\frac{\langle E(r)^p\rangle}{E(L)^p}=\lambda^{\gamma(p,\Delta)}, \quad (9)$$

which, given that the moments of $E(x,r)$ are independent of Δ, can be true only if $\gamma(p,\Delta)$ is Δ independent [13].

The recent data of PNP, however, show that the value of $\mu(2,\Delta)$ is very sensitive to the parameter Δ. For $\lambda=2$, their value of $\mu(2,1/2)$ is 0.15, but $\mu(2,0)$ is only 0.05. (We thank Gianni Pedrizzetti for sending us the numerical input data for Fig. 7 of PNP and the corresponding data for $\Delta=1/2$, which is not explicitly given in the published paper.) This qualitative effect is clearly seen in Fig. 10 of PNP, but we emphasize here that the effect is very large when expressed in terms of the scaling exponents. We were surprised to find the effect so large, so we examined the data of Sreenivasan and Stolovitzky used in [14] and found an effect of similar magnitude.

It follows from the previous discussion that either condition (i) or (ii) above must be violated [for if they held, $\gamma(p,\Delta)$ should be Δ independent]. In fact it was shown in [14] that multipliers at subsequent scales are not independent [were they independent, condition (ii) would obtain] and the direct violation of condition (ii) was established by PNP. Strict scale similarity is thus ruled out. Further, the dependence of the scaling exponents of the multipliers on Δ makes the connection between these exponents and the ones corresponding to dissipation even more tenuous. This suggests that the theory of multipliers is not as useful as once thought to explain the statistics of the dissipation.

To what extent can the theory of multipliers be used to explain issues regarding the dissipation? We shall see that there are situations in which the multipliers are useful tools, even though in these cases the argument is less direct than it would be if condition (ii) were fulfilled. One such situation is described in Sec. V in relation to the intermittency exponent. But first we discuss a few properties of multipliers.

III. FURTHER PROPERTIES OF THE MULTIPLIERS

In spite of the inability of the multipliers to describe several aspects of the dissipation statistics, the theory of multipliers is interesting in itself, both in the context of turbulent flows and in the context of stochastic processes in general. Thus it is worth understanding physically some of its properties, such as the statistical dependence of the multipliers, on the one hand, and the Δ dependence, on the other hand. We shall also show in this section that there is a relation between these two properties.

Let us first concentrate on the dependence between multipliers. In Ref. [14] it was shown from atmospheric turbulence data that for $\lambda=2$ and $|\Delta|=1/2$ (denoted in what follows by $\Delta=\pm 1/2$) a systematic statistical dependence exists between $M(r,2,\pm 1/2)$ and $M(2r,2,\pm 1/2)$. [For clarity in the notation, we keep track of the scale in the multiplier through the first argument in $M(r,\lambda,\Delta)$, even though as discussed in [14] the multiplier distribution is independent of r at the scales considered.] Specifically, the conditional PDF $P(M(r,2,\pm 1/2)|M(2r,2,\pm 1/2))$ for $0 \leqslant M(2r,2,\pm 1/2) \leqslant 1/2$ is narrower (has a smaller variance) than for $1/2 \leqslant M(2r,2,\pm 1/2) \leqslant 1$. This dependence is related to the more physical dependence of $M(r,2,\pm 1/2)$ on $E(2r,x)$, the latter being the total dissipation in the parent segment of length $2r$. One interesting feature noticed in [14] is that the conditional PDF of $M(r,2,\pm 1/2)$ given $E(2r,x)$ tends towards an asymptotic form for high values of $E(2r,x)$. Even though this feature was shown to hold experimentally only for $|\Delta|=1/2$ and $\lambda=2$, one can argue that this trend should be valid for arbitrary Δ and λ. An explanation for this can be given in terms of a similarity argument. Consider the conditional moments $\langle M(r,\lambda,\Delta)^q|\mathrm{Re}_r\rangle$ of the multipliers given the local Reynolds number $\mathrm{Re}_r = rE(r,x)^{1/3}/\nu$. It is reasonable to expect that for a given r, this statistics has to be universal (independent of viscosity) for sufficiently large local Reynolds number and therefore the Re_r dependence has to subside. But for a fixed scale, the local Reynolds number depends only on $E(r,x)$, and we conclude that the $E(r,x)$ dependence has to disappear when this quantity assumes large values. This feature of the multipliers is important in the analysis of the asymptotics of the scaling exponents of high-order moments of the dissipation, as will be seen in Sec. VI. An experimental assessment of this conjecture will have to await further work. The statistical dependence between multipliers was also demonstrated for the case $\Delta=0$ by PNP.

Let us now turn to the Δ dependence of the multipliers. This dependence was originally proposed by Novikov [4], who also explained its physical origin. Novikov's argument is worth recalling. Consider an interval of length $3r$ centered at x. The total dissipation in such interval is

$$E(3r,x)=E(r,x-r)+E(r,x)+E(r,x+r). \tag{10}$$

Since the dissipation autocorrelation function is observed [15] to decay monotonically with separation distance r, it follows that

$$\langle E(r,x-r)E(r,x)\rangle > \langle E(r,x-r)E(r,x+r)\rangle \tag{11}$$

and therefore

$$\langle E(3r,x)E(r,x)\rangle > \langle E(3r,x)E(r,x-r)\rangle. \tag{12}$$

Recalling that

$$E(r,x-r)=M(r,3,-1/2)E(3r,x) \tag{13}$$

and

$$E(r,x)=M(r,3,0)E(3r,x), \tag{14}$$

we obtain from the previous inequality that

$$\langle M(r,3,-1/2)E(3r,x)^2\rangle > \langle M(r,3,0)E(3r,x)^2\rangle, \tag{15}$$

from which it follows naturally that the statistics of $M(r,3,-1/2)$ and $M(r,3,0)$ are different, that is, there is a Δ dependence.

It is interesting to observe that any multiplier corresponding to a given Δ can be factorized as the product of two multipliers corresponding to $\Delta=\pm 1/2$. To perform this decomposition we introduce an intermediate stage in going from the big interval of length λr to the smaller interval of length r. For the intermediate interval, the right extreme coincides with the right extreme of the large segment and the left extreme coincides with the left extreme of the small interval. This intermediate segment has a length $\lambda_1 r$, where

$$\lambda_1=[(\lambda+1)\tfrac{1}{2}-(\lambda-1)\Delta]. \tag{16}$$

Then the multiplier linking the largest segment with the smallest one can be written as the product of the multiplier linking the largest segment and the intermediate one times the multiplier linking the intermediate segment with the smallest one. The first multiplier corresponds to $\Delta=1/2$, while the second one corresponds to $\Delta=-1/2$. Therefore one can in general write

$$M(r,\lambda,\Delta)=M(\lambda_1 r,\lambda/\lambda_1,1/2)M(r,\lambda_1,-1/2). \tag{17}$$

(It is clear that a similar factorization can be achieved if we go first to the left and then to the right.) Let us now assume that $M(\lambda_1 r,\lambda/\lambda_1,\pm 1/2)$ and $M(r,\lambda_1,\pm 1/2)$ are independent. Raising Eq. (17) to the power p, taking averages, using the independence between multipliers, assumption (i) of Sec. II, and local isotropy (which ensures that the statistics of $\Delta=1/2$ and $\Delta=-1/2$ are the same), we obtain that

$$\langle M(r,\lambda,\Delta)^p\rangle=\lambda^{\gamma(p,1/2)} \tag{18}$$

for any p, which shows that the moments of $M(r,\lambda,\Delta)$ and $M(r,\lambda,1/2)$ are the same for any Δ and any scale. The unavoidable conclusion is that independence between multipliers implies independence of Δ.

The picture that emerges is that dependence between multipliers and dependence on Δ are linked. Since the Δ dependence was shown to be expected from Novikov's arguments given above, it follows that a dependence between multipliers is to be expected. Furthermore, as discussed in Sec. II, a dependence between multipliers implies the breakdown of scale similarity.

IV. DEVIATIONS FROM SCALE SIMILARITY

If the scale r is in a universal inertial range, the moments of the multipliers do not depend on r, but dependence among the multipliers may lead to a deviation from scale similarity. To express this deviation, we generalize Eq. (4) to

$$\langle [M(r,\lambda,\Delta)]^p\rangle=\lambda^{\gamma(p,\lambda,\Delta)}, \tag{19}$$

where

$$\gamma(p,\lambda,\Delta)=\mu(p,\lambda,\Delta)-p, \tag{20}$$

and the exponents $\mu(p,\lambda,\Delta)$ are the same as given by PNP. The observed variation of these exponents with scale ratio λ for $\Delta=0$ is given in Fig. 7 of PNP. Using the same data set, we calculated the generalized dimension

$$D(p,\lambda,\Delta)=1-\frac{\mu(p,\lambda,\Delta)}{p-1} \tag{21}$$

as a function of p for various values of λ when $\Delta=1/2$. A significant dependence on scale ratio is observed for all values of p.

First we ask if this observed dependence is consistent with earlier published results. One of the tests done in the past regarding scale similarity was the check that the multiplier PDF for $\lambda=2$ did not depend on scale [14]. While this seems to be the case to a good approximation for inertial range scales, the strict scale similarity represented by Eq. (4) demands more stringent tests. Chhabra and Sreenivasan [8] studied $D(p,\lambda)$ for several values of λ. They were able to extract a scale ratio independent $f(\alpha)$ from their data, but they did this by adding a p-dependent prefactor to Eq. (19). Their curves for $D(p,\lambda)$ before this prefactor was extracted look qualitatively similar to those we obtained from the data of PNP [18]. The use of a p-dependent prefactor is equivalent to assuming a particular functional form for the dependence of the scaling exponents on scale ratio. It does not eliminate the deviations from scale similarity.

We next ask if there is any simple physically based way to model the dependence of the exponents on scale ratio. PNP have given an accurate empirical fit in which γ is a linear function of $\ln[\ln(\lambda)]$. An alternative possibility is that the dependence on scale ratio vanishes when the scale ratio becomes large. From Figs. 7(a) and 8(b) of [12] it is seen that the dependence of $\mu(p,\lambda,0)$ on λ does not vanish for large values of λ. The corresponding data for $\mu(p,\lambda,1/2)$ [19] show a similar behavior. The dependence on Δ remains strong even for large values of λ, however, and the exponents bear no simple relation to those for the dissipation. Thus, though it is tempting to assume that the statistics of the multipliers approaches the statistics of the dissipation when the scale ratio is large, there is no experimental information to support this assumption.

After having reviewed and presented some different material on properties of the multipliers, we now describe a few cases where one can establish a relation between the statistics of multipliers and that of the dissipation.

V. INTERMITTENCY EXPONENT

Since the work of Kolmogorov [1], the parameter used as the signature of small-scale intermittency is the so-called intermittency exponent μ, defined by

$$\langle E(r,x)^2\rangle=\langle\epsilon\rangle^2(r/L)^{2-\mu}. \tag{22}$$

The experimental situation for this exponent has been reviewed by Sreenivasan and Kailasnath [15], and the value of μ is typically about 0.25. Under the assumptions (i) and (ii) of Sec. II, one can compute this exponent from the theory of multipliers. It is worth recalling how this is done. As seen before, the assumption of independence makes the statistics of the multipliers independent of Δ. Let us choose a scale ratio λ and write $E(r)$, where $r=L/\lambda^k$ as a telescopic product of multipliers:

$$E(r,x)=E(L,x')M(L/\lambda,\lambda)M(L/\lambda^2,\lambda)\cdots M(L/\lambda^k,\lambda). \tag{23}$$

Raising Eq. (23) to the power p, using independence and scale similarity, and taking averages one finds that

$$\begin{aligned}\langle E(r,x)^p\rangle&=(L\langle\epsilon\rangle)^p\langle M(r,\lambda)^p\rangle^k\\&=(L\langle\epsilon\rangle)^p(r/L)^{-\log_\lambda\langle M(r,\lambda)^p\rangle},\end{aligned} \tag{24}$$

where we have used that $E(L,x')=L\epsilon_L(x')\approx L\langle\epsilon\rangle$. Comparing Eq. (24) with Eq. (22), we find that the intermittency exponent is given by

$$\mu=2+\log_\lambda\langle[M(r,\lambda)]^2\rangle. \tag{25}$$

Most measurements of μ from the multipliers have been for $\lambda=2$ and $\Delta=1/2$. Until recently these have also given $\mu\approx0.25$. However, as mentioned in Sec. II, the recent data of PNP show that the value of μ from Eq. (25) is very sensitive to the parameter Δ. For $\lambda=2$, their value of $\mu(2,1/2)$ is 0.15, but $\mu(2,0)$ is only 0.05. The ''universal'' intermittency exponent obtained from the multipliers is much smaller when the subinterval is centered on the larger interval than when it is at the end of the interval. This can be understood on the basis on the following nonrigorous argument. One would like to show, for example, that the variance of $M(r,2,-1/2)$ is greater than the variance of $M(r,2,0)$. This would be ensured if one shows that such is the case for the conditional second moment given the dissipation in the parent interval. To do this consider an interval of length $4r$ centered on x and divide this interval into four subintervals of length r centered on $x-3r/2$, $x-r/2$, $x+r/2$, and $x+3r/2$. The multiplier for $\Delta=-1/2$ corresponds to the sum of the first two subintervals and the multiplier for $\Delta=0$ corresponds to the sum of the second and third subintervals. The difference in the mean-square multipliers for these two cases is given by

$$\begin{aligned}&\left\langle\left(\frac{E(r,x-3r/2)+E(r,x-r/2)}{E(4r,x)}\right)^2\middle|E(4r,x)\right\rangle\\&\quad-\left\langle\left(\frac{E(r,x-r/2)+E(r,x+r/2)}{E(4r,x)}\right)^2\middle|E(4r,x)\right\rangle.\end{aligned} \tag{26}$$

If we expand the above expression and add and subtract the dissipation in the fourth subinterval, the desired inequality can be rewritten in the form

$$\frac{1}{E(4r,x)}[\langle E(r,x-3r/2)|E(4r,x)\rangle - \langle E(r,x+r/2)|E(4r,x)\rangle] + \frac{1}{E(4r,x)^2}[\langle E(r,x-3r/2)E(r,x-r/2)|E(4r,x)\rangle - \langle E(r,x-r/2)E(r,x+r/2)|E(4r,x)\rangle] + \frac{1}{E(4r,x)^2}[\langle E(r,x+r/2)E(r,x+3r/2)|E(4r,x)\rangle - \langle E(r,x-3r/2)E(r,x+3r/2)|E(4r,x)\rangle]>0. \quad (27)$$

The difference in the first set of square brackets is likely to be close to zero, as is the difference in the second set of square brackets, the latter because they are correlations between nearest neighbors. However, the difference in the third set of square brackets is most likely to be positive because it is the difference between the correlation of nearest neighbors and second-nearest neighbors. If this is the case, the direction of the previous inequality is justified and therefore the variance of $M(r,2,-1/2)$ is greater than the variance of $M(r,2,0)$.

Now, why is the intermittency exponent computed from the multipliers corresponding to $\Delta=1/2$ so suspiciously close to the μ measured directly, whereas the case $\Delta=0$ gives a result that is quite different? We now show that there are strong reasons why the case $\Delta=1/2$ is the one that gives the result closest to the one computed from the dissipation. Noting that [16]

$$\langle \epsilon(x+r)\epsilon(x)\rangle = \frac{1}{2}\frac{d^2}{dr^2}\langle E(r,x)^2\rangle \quad (28)$$

and using (22), it is clear that the intermittency exponent determines the correlation between $\epsilon(x+r)$ and $\epsilon(x)$ through

$$\langle \epsilon(x+r)\epsilon(x)\rangle = A\langle\epsilon\rangle^2\left(\frac{r}{L}\right)^{-\mu}, \quad (29)$$

where $A=(2-\mu)(1-\mu)/2$. It can be easily shown from Eqs. (1) and (29) that

$$\langle E(r,x-l/2)E(l,x+r/2)\rangle = \frac{A}{(2-\mu)(1-\mu)}(L\langle\epsilon\rangle)^2 \times\left[\left(\frac{r+l}{L}\right)^{2-\mu} - \left(\frac{l}{L}\right)^{2-\mu} - \left(\frac{r}{L}\right)^{2-\mu}\right]. \quad (30)$$

On the other hand, given that

$$E(r+l,x)=E(r,x-l/2)+E(l,x+r/2) \quad (31)$$

and

$$E(r,x-l/2)=M(r,\lambda,-1/2)E(r+l,x), \quad (32)$$

where $\lambda=(r+l)/r$ as usual, we obtain

$$\langle E(r,x-l/2)E(l,x+r/2)\rangle = \langle [M(r,\lambda,-1/2)-M(r,\lambda,-1/2)^2]E(r+l,x)^2\rangle. \quad (33)$$

Assuming *only at this point* that $M(r,\lambda,-1/2)$ and $M(r,\lambda,-1/2)^2$ are not correlated with $E(r+l,x)$, we obtain from Eq. (33) that

$$\langle E(r,x-l/2)E(l,x+r/2)\rangle = (L\langle\epsilon\rangle)^2\left(\frac{r+l}{L}\right)^{2-\mu}[\langle M(r,\lambda,-1/2)\rangle - \langle M(r,\lambda,-1/2)^2\rangle]. \quad (34)$$

Using that $\langle M(r,\lambda,-1/2)\rangle = 1/\lambda$ [valid under the assumption of decorrelation between $M(r,\lambda,-1/2)$ and $E(r+l,x)$] and equating Eqs. (30) and (34), we obtain

$$\langle M(r,\lambda,-1/2)^2\rangle = \frac{1}{\lambda}-\frac{1}{2}+\frac{1}{2}\left[\left(1-\frac{1}{\lambda}\right)^{2-\mu}+\left(\frac{1}{\lambda}\right)^{2-\mu}\right], \quad (35)$$

which coincides with Eq. (25) only for $\lambda=2$. Stated briefly, the only scale ratio for which the intermittency exponent can be found without contradictions is $\lambda=2$ and $\Delta=1/2$, which is the reason why the intermittency exponent computed for these values of λ and Δ is close to the one computed directly from the dissipation.

VI. BEHAVIOR OF HIGH MOMENTS

One of the important results of [9] and PNP concerns the asymptotic behavior of the scaling exponents $\mu(p,\Delta)$ for large values of p, which will be called here Novikov's gap theorem. For completeness, let us rederive this theorem for the multipliers (as opposed to for the breakdown coefficients). Our starting point is the definition of the scaling exponents for the multipliers [see Eq. (4)]

$$\gamma(q,\Delta) = -\log_\lambda\langle M(r,\lambda,\Delta)^q\rangle = -\log_\lambda\left[\int_0^1 M(r,\lambda,\Delta)^q P(M;r,\lambda,\Delta)\,dM\right]. \quad (36)$$

Now assume that there is a gap in the multiplier distribution, that is,

$$\int_m^1 P(M;r,\lambda,\Delta)\,dM = 0, \quad (37)$$

where $0<m<1$. Then, from Eq. (36) one finds that

$$\gamma(q,\Delta)\geqslant q\log_\lambda(1/m) \quad (38)$$

and

$$\xi \equiv \lim_{q\to\infty}\frac{\gamma(q,\Delta)}{q}\geqslant \log_\lambda(1/m)>0. \quad (39)$$

However, if there is no gap $m=1$ and one has

$$0 \leqslant \xi \leqslant \lim_{q\to\infty}\left(-\frac{1}{q}\right)\log_{\lambda}\left[(m^*)^q\int_{m^*}^{1}P(M;r,\lambda,\Delta)dM\right]$$
$$=\log_{\lambda}(1/m^*) \tag{40}$$

for any m^*. Thus $\xi=0$.

If the multipliers were independent at successive scales, then the previous result would imply that, if there is no gap (as the experimental evidence suggests),

$$\lim_{q\to\infty}\frac{\log_{r/L}[\langle E(r,x)^q\rangle/(L\langle\epsilon\rangle)^q]}{q}=0. \tag{41}$$

This is Novikov's result [9], which purports to show that the She-Leveque model [10] is inconsistent with experiment, as it predicts that the right-hand side of (41) is 1/3 instead of 0. However, as we have seen, the multipliers are dependent, and this obscures any inference from the world of multipliers to that of dissipation.

In order to take into account these cumbersome dependences, it is necessary to use conditional statistics. Basically, we are interested in the ratio $\langle E(r,x)^q\rangle/\langle E(l,x)^q\rangle$, which can be rewritten in terms of the multipliers as

$$\frac{\langle E(r)^q\rangle}{\langle E(l)^q\rangle}=\frac{1}{\langle E(l)^q\rangle}\left\langle\frac{E(r)^q}{E(l)^q}[E(l)^q]\right\rangle$$
$$=\frac{1}{\langle E(l)^q\rangle}\left\langle\left\langle\frac{E(r)^q}{E(l)^q}\middle|E(l)\right\rangle E(l)^q\right\rangle. \tag{42}$$

For the remainder of this section we suppress the second argument in $E(r,x)$ for notational simplicity since it plays no role. We recognize in the previous equation that $E(r)/E(l)=M(r,l/r,\Delta)$. Therefore we have that

$$\frac{\langle E(r)^q\rangle}{\langle E(l)^q\rangle}=\int_0^\infty\langle M(r,l/r,\Delta)^q|E(l)\rangle D(E(l))\frac{E(l)^q}{\langle E(l)^q\rangle}dE(l), \tag{43}$$

where $D(E(l))$ is the PDF of $E(l)$. We are interested in the limit of q tending to ∞. In this limit, the kernel of the integral

$$Q(E(l),q)=D(E(l))E(l)^q/\langle E(l)^q\rangle \tag{44}$$

can be interpreted as a PDF since it is positive and integrates to 1. It selects increasingly large values of $E(l)$ as q becomes larger. On the other hand, it is found from the analysis of experimental data [14] that the conditional distribution of $M(r,2,\pm 1/2)$, given $E(l)$, becomes independent of $E(l)$ when this quantity is large enough. As we discussed in Sec. III, a similarity argument suggests the conjecture that the same result should hold valid for arbitrary λ and Δ, namely, that the statistics of $M(r,\lambda,\Delta)$, given $E(l)$, becomes independent of $E(l)$ for sufficiently large $E(l)$. Therefore, as q increases, Eq. (43) tends to

$$\langle E(r)^q\rangle/\langle E(l)^q\rangle=\langle M(r,l/r,\Delta)^q|E(l)\rangle \gg l\langle\epsilon\rangle. \tag{45}$$

(Notice that we have used that $\langle M^q|E\rangle$ tends to be independent of E for large E.) Now, we have seen when we discussed Novikov's gap theorem in the beginning of this section that for a random variable M bounded between 0 and 1 and whose distribution has no gap, $\langle M^q\rangle\sim\lambda^{\gamma(q)}$, with $\gamma(q)/q\to 0$ for high q. This means that if $\langle M^q\rangle$ scales with λ, this scaling is at most sublinear. Therefore, recalling that $E(r)=r\epsilon_r$ [see Eq. (1)], it follows from (45) that the asymptotic behavior of $\langle\epsilon_r^q\rangle/\langle\epsilon_l^q\rangle$ for sufficiently large q should be

$$\frac{\langle\epsilon_r^q\rangle}{\langle\epsilon_l^q\rangle}=\left(\frac{l}{r}\right)^q\langle M(r,\lambda,\Delta)^q|E(l)\to\infty\rangle\sim\lambda^{\gamma(q)-q}\underset{q\gg 1}{\sim}\lambda^{-q}. \tag{46}$$

This line of argument leads to basically the same conclusion reached by Novikov, namely, the scaling of $\langle\epsilon_r^q\rangle\sim r^{-q}$, and not to $\langle\epsilon_r^q\rangle\sim r^{-hq}$ with $0<h<1$ as proposed for example in the phenomenology of She and Leveque [10]. The previous line of reasoning has a caveat, however. In Eq. (45), the left-hand side is independent of Δ, whereas the right-hand side is not, at least *a priori*. In order for our reasoning to be consistent, one should have that $\langle M(r,l/r,\Delta)^q|l\epsilon_l(x)\rangle \gg l\langle\epsilon\rangle$ be independent of Δ for sufficiently large q. This is a prediction that has to be checked experimentally. In any case, it is interesting to notice that this independence, if true, would imply some constraints on the structures responsible for the most intense events.

VII. DISCUSSION AND CONCLUSIONS

We have reviewed the statistical properties of the random multipliers under conditions of strict scale similarity. We started from the observation of PNP [12] that these properties are sensitive to the parameter Δ, which describes the position of the subinterval of length r in the larger interval of length λr. This observation is in agreement with the original suggestion by Novikov [4], which we have reviewed in Sec. III. We then pointed out that this dependence on Δ strongly suggests that the multipliers for successive cascade steps must be statistically dependent, thus violating the conditions of strict scale similarity. In Sec. IV, we reviewed the direct evidence for deviations from strict scale similarity and pointed out that this evidence is not in conflict with earlier experiments.

In Sec. V, we examined the evidence on the ''universal'' intermittency exponent μ and gave a physical argument in support of the observation that this exponent should be smaller for $\Delta=0$ when the subinterval is centered on the larger interval than for $\Delta=1/2$, when it is at one end of the larger interval. We also suggested that it is no accident that earlier measurements for $\Delta=1/2$ and $\lambda=2$ gave a value of μ from the multipliers close to that observed directly from the statistics of the turbulent energy dissipation. In Sec. VI, we considered the behavior of high moments of the multiplier and the corresponding asymptotic behavior of the scaling exponents. We started from the observation [14] that the PDF of the multiplier, conditioned on the total dissipation in the parent interval, appears to be independent of this total dissipation when this quantity becomes large. From this observation, we were able to extend Novikov's ''gap theorem'' [9] to the case when the multipliers are statistically dependent. This theorem constrains the behavior of the scaling exponents of the multipliers to increase less rapidly than linearly at high orders.

Although the statistics of the multipliers remain an interesting universal inertial range property of high Reynolds number turbulence, the arguments that we have given suggest that they bear no simple relation to the statistics of the local dissipation. In particular, we have shown that the deviations from strict scale similarity, observed by PNP [12], are to be expected theoretically. These deviations suggest that the application of the theory of infinitely divisible distributions to the multipliers by Novikov [9] is not appropriate since this theory assumes independence between multipliers. We also suggested that Novikov's gap theorem, which is correct for the multipliers, cannot easily be extended to the dissipation (or to the moments of velocity differences) without some subtle precautions. If our conjecture (based on some experimental data and a similarity argument) that $\langle M(r,\lambda,\Delta)^q | E(l)\rangle$ tends to be independent of $E(l)$ for high values of $E(l)$ holds, Novikov's suggestion that his gap theorem constrains popular models [10] for the scaling of the moments of the dissipation remains valid.

What is the connection between our results and the usual multifractal description of turbulent dissipation [17]? As discussed, for example, in [2], the multifractal picture follows naturally from a multiplicative random process of independent multipliers. Thus, from the knowledge of the multiplier PDF and the assumption of independent, scale invariant multipliers, the multifractal spectrum can be easily derived. However, we discussed in Sec. III that the multipliers are not independent and then it is not at all clear which of the properties of the multifractal spectrum derived from the assumption of independence survive. This issue was considered in Ref. [14], where it was suggested that the nature of the dependence is such that only the region of the multifractal spectrum related to the negative moments of the dissipation is expected to differ noticeably from the equivalent region of the spectrum computed assuming independence. On the other hand, the part of the spectrum associated with the positive moments computed from the multiplier PDF assuming independence is expected to coincide with the spectrum computed directly, without the use of multipliers. It is in this latter sense that the dependence between multipliers was deemed benign in Ref. [14]. Finally, it is worth emphasizing that it might be possible to generate a multifractal spectrum without an underlying random multiplicative process, and even if such a process exists, it need not be characterized by the type of multipliers that we have studied here. Thus even though the ''classical'' description of scale similar multipliers has severe limitations, these limitations do not preclude the validity of the multifractal picture of turbulent energy dissipation.

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